$$\frac{d^2\psi(z)}{dz^2} - \frac{2m}{\hbar^2}(mgz-E)\psi(z)=0$$

$$x = \left(\frac{\hbar^2}{2m^2g}\right)^{2/3}\left(\frac{2m}{\hbar^2}\right)(mgz-E)$$

$$\Rightarrow \frac{d^2\phi(x)}{dx^2} - x\,\phi(x)=0 \qquad \begin{array}{l}\phi(0)=0\\ x\in[0,\infty]\end{array}$$

$$\phi(x) = B\,\underbrace{\frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3}+xt\right)dt}_{Ai(x)}$$

Sinir şartları

$$\psi(0)=0$$

$$z=0 \Rightarrow x=\left(\frac{2}{mg^2\hbar^2}\right)^{1/3}$$

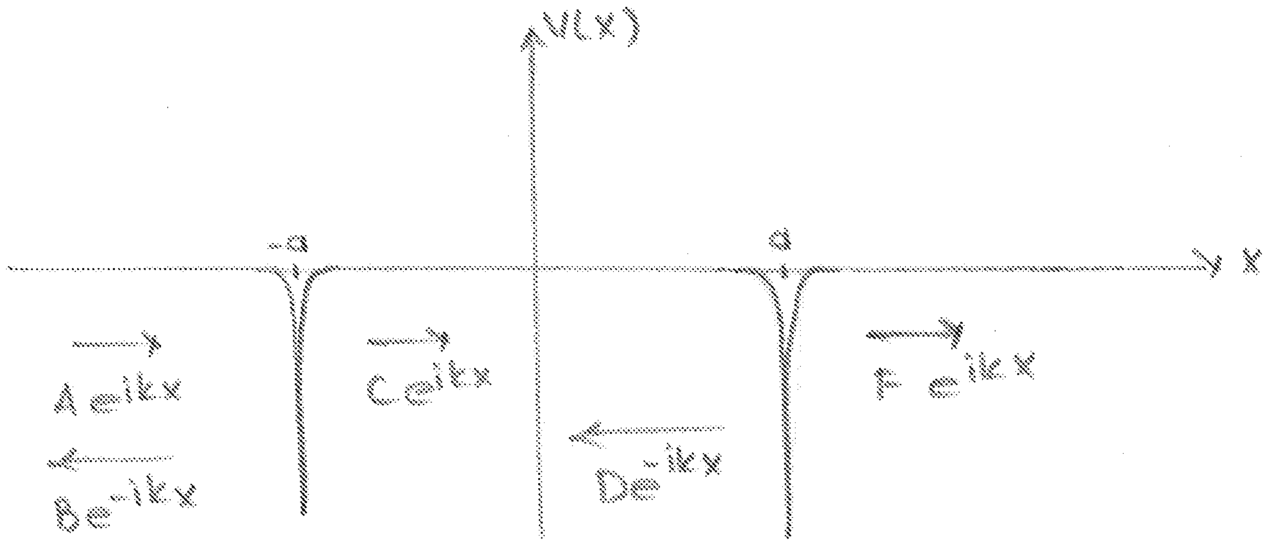
$$\psi(0)=0 \Rightarrow \phi\left[-\left(\frac{2}{mg^2\hbar^2}\right)^{1/3}E\right]=0$$

$$\Rightarrow Ai\Big[\underbrace{-\left(\frac{2}{mg^2\hbar^2}\right)^{1/3}E}_{R_n}\Big]=0 \Rightarrow -\left(\frac{2}{mg^2\hbar^2}\right)^{1/3}E_n=R_n$$

Cevap 2)

$V(x) = -\alpha[\delta(x-a) + \delta(x+a)]$ potansiyeli için

iletim katsayısını hesaplayın (α pozitif gerçel sayı)



$\psi_I(x) = Ae^{ikx} + Be^{-ikx}$ $\quad x < -a$

$\psi_{II}(x) = Ce^{ikx} + De^{-ikx}$ $\quad -a < x < a$

$\psi_{III}(x) = Fe^{ikx}$ $\quad x > a$

1.) Dalga fonksiyonu heryerde sürekli olmalı

2.) $\frac{d\psi}{dx}$, potansiyelin sonsuz olduğu yerler dışında sürekli olmalı.

Ψ dalga fonksiyonunun $-a$ da sürekli olması için,

$$\Psi_I(-a) = \Psi_{II}(-a)$$

$$A e^{-ika} + B e^{ika} = C e^{-ika} + D e^{ika}$$

$$e^{-2ika} A + B = e^{-2ika} C + D \quad \ldots \ldots (1)$$

a da sürekli olması için,

$$\Psi_{II}(a) = \Psi_{III}(a)$$

$$C e^{ika} + D e^{-ika} = F e^{ika} \rightarrow C + e^{-2ika} D = F \quad \ldots (2)$$

$$\lim_{x \to -a} \frac{d\Psi_I}{dx} = \lim_{x \to -a} \frac{d\Psi_{II}}{dx}$$ sınır şartı yeni bir bilgi vermiyor

δ delta potansiyeli $-a$ ve a daki süreksizliği belirleyecek,

Schrödinger denkleminin $-a-\epsilon$'dan $-a+\epsilon$'a kadar integralini alıp $\epsilon \to 0$ giderken limitini alalım:

$$-\frac{\hbar^2}{2m} \int_{-a-\epsilon}^{-a+\epsilon} \frac{d^2\Psi}{dx^2} dx + \int_{-a-\epsilon}^{-a+\epsilon} V(x)\Psi(x)\, dx = E \underbrace{\int_{-a-\epsilon}^{-a+\epsilon} \Psi(x)\, dx}_{0}$$

$$-\frac{\hbar^2}{2m}\left(\frac{d\psi}{dx}\right)\Bigg|_{-a-\epsilon}^{-a+\epsilon} + \int_{-a-\epsilon}^{-a+\epsilon} -\alpha\left(\delta(x-a)+\delta(x+a)\right)\psi(x)\,dx = 0$$

$$\left(\frac{d\psi}{dx}\right)\Bigg|_{-a-\epsilon}^{-a+\epsilon} + \frac{2m\alpha}{\hbar^2}\left(\int_{-a-\epsilon}^{-a+\epsilon}\delta(x-a)\psi(x)\,dx + \int_{-a-\epsilon}^{-a+\epsilon}\delta(x+a)\psi(x)\,dx\right) = 0$$

$$\lim_{\epsilon\to 0}\left(\frac{d\psi}{dx}\right)\Bigg|_{-a-\epsilon}^{-a+\epsilon} = -\frac{2m\alpha}{\hbar^2}\left(\cancelto{0}{\lim_{\epsilon\to 0}\int_{-a-\epsilon}^{-a+\epsilon}\delta(x-a)\psi(x)\,dx} + \lim_{\epsilon\to 0}\int_{-a-\epsilon}^{-a+\epsilon}\delta(x+a)\psi(x)\,dx\right)$$

$$\Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\left(\psi(-a)\right) \qquad (3)$$

$$x<-a \Rightarrow \frac{d\psi}{dx} = ikAe^{ikx} - ikBe^{-ikx}$$

$$\lim_{x\to -a^-}\frac{d\psi}{dx} = ikAe^{-ika} - ikBe^{+ika}$$

$$x>-a \Rightarrow \frac{d\psi}{dx} = ikCe^{ikx} - ikDe^{-ikx}$$

$$\lim_{x\to -a^+}\frac{d\psi}{dx} = ikCe^{-ika} - ikDe^{+ika} \Rightarrow$$

$$\Delta\left(\frac{d\psi}{dx}\right) = \lim_{x\to -a^+} \frac{d\psi}{dx} - \lim_{x\to -a^-} \frac{d\psi}{dx}$$

$$= ik\left(Ce^{-ika} - De^{+ika}\right) - ik\left(Ae^{-ika} - Be^{+ika}\right) \quad \ldots\ldots (4)$$

(3) ile (4) ü eşitlersek,

$$\Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\left(Ae^{-ika} + Be^{ika}\right)$$

$$= ik\left(Ce^{-ika} - De^{ika}\right) - ik\left(Ae^{-ika} - Be^{ika}\right)$$

$$e^{-2ika}C - D = e^{-2ika}\left(\frac{i2m\alpha}{\hbar^2 k} + 1\right)A + \left(\frac{i2m\alpha}{\hbar^2 k} - 1\right)B$$

$$\ldots\ldots (5)$$

Şimdi de Sch. denkleminin $+a-\epsilon$'dan $+a-\epsilon$'a kadar integralini alıp $\epsilon \to 0$ giderken limitini alalım,

$$\Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\left(\psi(a)\right) \quad \ldots \quad (6)$$

$$= -\frac{2m\alpha}{\hbar^2}\left(Fe^{ika}\right)$$

$$x < a \rightarrow \frac{d\psi}{dx} = ikC^{ikx} - ikDe^{-ikx}$$

$$\lim_{x \rightarrow a^-} \frac{d\psi}{dx} = ikCe^{ika} - ikDe^{-ika}.$$

$$x > a \rightarrow \frac{d\psi}{dx} = ikFe^{ikx}$$

$$\lim_{x \rightarrow a^+} \frac{d\psi}{dx} = ikFe^{ika}$$

$$\Delta\left(\frac{d\psi}{dx}\right) = \lim_{x \rightarrow a^+} \frac{d\psi}{dx} - \lim_{x \rightarrow a^-} \frac{d\psi}{dx}$$

$$= ikFe^{ika} - ik\left(Ce^{ika} - De^{-ika}\right)$$

$$= -\frac{2m\alpha}{\hbar^2}\left(Fe^{ika}\right)$$

$$C - e^{-2ika}D = \left(1 - \frac{i2m\alpha}{\hbar^2 k}\right)F \quad \ldots \quad (7)$$

$$C + e^{-2ika}D = F \quad \ldots \quad (2) \quad \text{idi.}$$

(2) ile (7) yi toplarsak,

$$2C = \left(2 - \frac{i2m\alpha}{\hbar^2 k}\right)F \quad , \quad \gamma = \frac{i2m\alpha}{\hbar^2 k} \text{ olsun}$$

$$2C = \left(2 - \gamma\right)F \quad \ldots \quad (8)$$

(2) ile (7) yi çıkarırsak

$2e^{-2ika} D = \gamma F$, $e^{-2ika} = \beta$ olsun

$2\beta D = \gamma F \Rightarrow 2D = \left(\frac{\gamma}{\beta}\right) F$ . . . . (9)

(1) ile (5) i hatırlayalım,

$\beta A + B = \beta C + D$ . . . . (1)

$\beta C - D = \beta(\gamma + 1) A + (\gamma - 1) B$ . . . (5) idi

(1) ile (5) i toplarsak,

$2\beta C = \beta A + B + \beta(\gamma+1) A + (\gamma - 1) B$

$= \beta A + \beta(\gamma+1) A + \gamma B \Rightarrow$

$2C = (\gamma + 2) A + (\gamma/\beta) B$ . . . . . (10)

(1) ile (5) i çıkaralım,

$2D = \beta A + B - \beta(\gamma+1) A - (\gamma - 1) B \Rightarrow$

$2D = -\gamma\beta A + (2 - \gamma) B$ . . . . . (11)

(8) ile (10), (9) ile (11) eşit old. göre,

$(2 - \gamma) F = (\gamma + 2) A + (\gamma/\beta) B$ . . . (12)

$\left(\frac{\gamma}{\beta}\right) F = -\gamma\beta A + (2 - \gamma) B$ . . . . (13) old. görülür.

(Amacımız iletim katsayısı $T = \left|\frac{F}{A}\right|^2$ yi bulmak.)

(12)'yi $\beta(2-\gamma)$ ile, (13)'ü de γ ile

çarpıp, birbirinden çıkarırsak,

$$\beta(2-\gamma)^2 F = \beta(4-\gamma^2)A + \cancel{\gamma(2-\gamma)B}$$

$$\left(\frac{\gamma^2}{\beta}\right)F = -\gamma^2\beta A + \cancel{\gamma(2-\gamma)B}$$

$$\left(\beta(2-\gamma)^2 - \frac{\gamma^2}{\beta}\right)F = \beta\left(4 - \cancel{\gamma^2} + \cancel{\gamma^2}\right)A \Rightarrow$$

$$\frac{F}{A} = \frac{4}{(2-\gamma)^2 - (\gamma^2/\beta^2)}$$ bulunur.

$$\gamma = \frac{i\,2m\alpha}{\hbar^2 k}, \quad \beta = e^{-2ika}$$

$$T = \left|\frac{F}{A}\right|^2 = \left|\frac{4}{(2-\gamma)^2 - (\gamma^2/\beta^2)}\right|^2$$

$\Rightarrow$ //

(cevap 3) $|\alpha\rangle = 5i|\phi_1\rangle + |\phi_2\rangle$

$|\beta\rangle = i|\phi_1\rangle - 5i|\phi_2\rangle$

a) $\langle \alpha + \beta| = ?$

$\langle \alpha| = -5i\langle \phi_1| + \langle \phi_2|$

$\langle \beta| = -i\langle \phi_1| + 5i\langle \phi_2|$

$\langle \alpha + \beta| = -6i\langle \phi_1| + (5i+1)\langle \phi_2|$

b) $\langle \alpha|\beta\rangle = ?$, $\langle \phi_i|\phi_j\rangle = \delta_{ij}$

$$\langle \alpha|\beta\rangle = (-5i\langle \phi_1| + \langle \phi_2|)(i|\phi_1\rangle - 5i|\phi_2\rangle)$$

$$= 5\underbrace{\langle \phi_1|\phi_1\rangle}_{1} - 25\underbrace{\langle \phi_1|\phi_2\rangle}_{0} + i\underbrace{\langle \phi_2|\phi_1\rangle}_{0} - 5i\underbrace{\langle \phi_2|\phi_2\rangle}_{1}$$

$$= 5 - 5i$$

$$\langle \beta|\alpha\rangle = (-i\langle \phi_1| + 5i\langle \phi_2|)(5i|\phi_1\rangle + |\phi_2\rangle) = 5 + 5i$$

c) Schwarz eşitsizliği,

$$|\langle \alpha|\beta\rangle|^2 \leq \langle \alpha|\alpha\rangle\langle \beta|\beta\rangle$$

$$\langle \alpha|\alpha\rangle = (-5i\langle \phi_1| + \langle \phi_2|)(5i|\phi_1\rangle + |\phi_2\rangle$$

$$= 25\underbrace{\langle \phi_1|\phi_1\rangle}_{1} - 5i\underbrace{\langle \phi_1|\phi_2\rangle}_{0} + 5i\underbrace{\langle \phi_2|\phi_1\rangle}_{0} + \underbrace{\langle \phi_2|\phi_2\rangle}_{1}$$

$$= 25 + 1$$

$$= 26$$

$$\langle \beta|\beta\rangle = (-i\langle \phi_1| + 5i\langle \phi_2|)(i|\phi_1\rangle - 5i|\phi_2\rangle)$$

$$= \langle \phi_1|\phi_1\rangle - 5\underbrace{\langle \phi_1|\phi_2\rangle}_{0} - 5\underbrace{\langle \phi_2|\phi_1\rangle}_{0} + 25\langle \phi_2|\phi_2\rangle$$

$$= 1 + 25 = 26 \qquad \Rightarrow$$

$$|\langle\alpha|\beta\rangle|^2 = \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle$$
$$= (5-5i)(5+5i) = 25 + \cancel{25i} - \cancel{25i} + 25$$
$$= 50$$

$|\langle\alpha|\beta\rangle|^2 \leq \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$ Schwarz eşitsizliğinin

$50 \leq 26 \cdot 26$

sağlandığı görülür.

d) Üçgen eşitsizliği,

$$\| |\alpha\rangle + |\beta\rangle \| \leq \|\alpha\| + \|\beta\|$$

$$\| |\alpha\rangle + |\beta\rangle \| = \sqrt{(\langle\alpha| + \langle\beta|)(|\alpha\rangle + |\beta\rangle)}$$
$$= \left[(-6i\langle\Phi_1| + (1+5i)\langle\Phi_2|)(6i|\Phi_1\rangle + (1-5i)|\Phi_2\rangle)\right]^{1/2}$$
$$= (36 + (1+5i)(1-5i))^{1/2}$$
$$= (36 + (1-25i^2))^{1/2}$$
$$= (36 + 1 + 25)^{1/2}$$
$$= \sqrt{62}$$

$$\|\alpha\| = \sqrt{\langle\alpha|\alpha\rangle} = \sqrt{26}$$
$$\|\beta\| = \sqrt{\langle\beta|\beta\rangle} = \sqrt{26}$$

$\| |\alpha\rangle + |\beta\rangle \| \leq \|\alpha\| + \|\beta\|$ Üçgen eşitsizliğinin

$\sqrt{62} \leq \sqrt{26} + \sqrt{26}$ sağlandığı görülür.

Soru 4)

a) $(A + A^{\dagger})^{\dagger} = A^{\dagger} + (A^{\dagger})^{\dagger} = A^{\dagger} + A$

$(A + A^{\dagger})^{\dagger} = (A + A^{\dagger})$ olduğuna göre $(A + A^{\dagger})$ operatörü hermityendir.

$(i(A + A^{\dagger}))^{\dagger} = -i(A^{\dagger} + A)$ old. göre $i(A + A^{\dagger})$ operatörü hermityen değildir.

$(i(A - A^{\dagger}))^{\dagger} = -i(A^{\dagger} - A) = i(A - A^{\dagger})$ operatörü hermityendir.

$(\{A, A^{\dagger}\}^{\dagger})^{\dagger} = \{A, A^{\dagger}\} = AA^{\dagger} + A^{\dagger}A$

$\{A, A^{\dagger}\}^{\dagger} = (AA^{\dagger} + A^{\dagger}A)^{\dagger} = AA^{\dagger} + A^{\dagger}A$

$\{A, A^{\dagger}\}^{\dagger} = (\{A, A^{\dagger}\}^{\dagger})^{\dagger}$ olduğuna göre $\{A, A^{\dagger}\}^{\dagger}$ operatörü hermityendir.

Not : $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ olduğundan $(AA^{\dagger})^{\dagger} = (A^{\dagger})^{\dagger}A^{\dagger}$
$= AA^{\dagger}$

4b) $e^{A} e^{B} = e^{A+B} e^{[A,B]/2}$

$$\left[A^0 + A + \frac{A^2}{2!} + \cdots\right]\cdot\left[B^0 + B + \frac{B^2}{2!} + \cdots\right] = e^{A}e^{B}$$

$$= \left[1 + B + \frac{B^2}{2!} + A + AB + \frac{AB^2}{2!} + \frac{A^2}{2!} + \frac{A^2}{2!}B + \frac{A^2B^2}{2!2!}\right]$$

$$= \left[1 + A + B + \frac{A^2}{2} + \frac{B^2}{2} + \frac{AB}{2} + \frac{BA}{2} + \frac{AB}{2} - \frac{BA}{2}\right.$$
$$\left. + \frac{AB^2}{2} + \frac{A^2B}{2} + \frac{A^2B^2}{2!2!} + \cdots\right]$$

$$= \left[1 + \frac{AB}{2} - \frac{BA}{2} + \cdots + A + \frac{A^2B}{2} - \frac{ABA}{2} + \cdots + B\right.$$
$$+ \frac{BAB}{2} - \frac{B^2A}{2} + \cdots + \frac{A^2}{2} + \frac{A^2AB}{2} + \cdots$$
$$\left. + \frac{B^2}{2} + \frac{B^2AB}{2} + \cdots + \frac{AB}{2} + \cdots + \frac{BA}{2} + \cdots\right]$$

$$= \left[1 + A + B + \frac{A^2}{2} + \frac{B^2}{2} + \frac{AB}{2} + \frac{BA}{2} + \cdots\right]$$
$$\times\left[1 + \frac{AB}{2} - \frac{BA}{2} + \cdots\right]$$

$$= \left[1 + (A+B) + \frac{(A+B)^2}{2!} + \cdots\right] \times \left[1 + \left(\frac{[A,B]}{2}\right)^1 + \left(\frac{[A,B]}{2}\right)^2 + \cdots\right]$$

$$= e^{A+B} e^{[A,B]/2}$$

4c) Baker-Hausdoff lemma ifadesini türetelim;

$$e^{A}Be^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots$$

$$e^{A}Be^{-A} = \left(1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots\right)B\left(1 - A + \frac{A^2}{2!} - \frac{A^3}{3!} + \dots\right)$$

$$= \left(1 + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots\right)\left(B - BA + \frac{BA^2}{2!} - \frac{BA^3}{3!} + \dots\right)$$

$$= B - BA + \frac{BA^2}{2} - \frac{BA^3}{3!} + AB - ABA + \frac{ABA^2}{2} - \frac{ABA^3}{3!}$$

$$+ \frac{A^2}{2}B - \frac{A^2}{2}BA + \frac{A^2BA^2}{4} - \frac{A^2BA^3}{2!3!} + \frac{A^3}{3!}B$$

$$- \frac{A^3}{3!}BA + \frac{A^3}{3!}\frac{BA^2}{2!} - \frac{A^3}{3!}\frac{BA^3}{3!} + \dots$$

$$= B + (AB - BA) + \frac{1}{2}(AAB - ABA - ABA + BAA)$$

$A[A,B] - [A,B]A$

$$+ \frac{1}{3!}(A^3B - 3A^2BA + 3ABA^2 - BA^3) + \dots$$

$$= B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots$$

Cevap 5)

a) $[A,BC] = [A,B]C + B[A,C]$ olduğunu gösterelim.

$$[A,BC] = ABC - BCA$$

$$\begin{aligned}[A,B]C + B[A,C] &= (AB-BA)C + B(AC-CA) \\ &= ABC - \cancel{BA}C + \cancel{BA}C - BCA \\ &= ABC - BCA\end{aligned}$$

$$[A,BC] = [A,B]C + B[A,C]$$

b) $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ old. göst.

$$\begin{aligned}[A,[B,C]] &= A[B,C] - [B,C]A \\ &= A(BC-CB) - (BC-CB)A \\ &= \cancel{ABC} - \cancel{ACB} - \cancel{BCA} + \cancel{CBA} \quad \dots\dots (1)\end{aligned}$$

$$\begin{aligned}[B,[C,A]] &= B[C,A] - [C,A]B \\ &= B(CA-AC) - (CA-AC)B \\ &= \cancel{BCA} - \cancel{BAC} - \cancel{CAB} + \cancel{ACB} \quad \dots\dots (2)\end{aligned}$$

$$\begin{aligned}[C,[A,B]] &= C[A,B] - [A,B]C \\ &= C(AB-BA) - (AB-BA)C \\ &= \cancel{CAB} - \cancel{CBA} - \cancel{ABC} + \cancel{BAC} \quad \dots\dots (3)\end{aligned}$$

$(1) + (2) + (3) = 0$ olur.

c) $[A,B]^\dagger = [B^\dagger, A^\dagger]$ old. göst.

$[A,B]^\dagger = (AB - BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger$

$[B^\dagger, A^\dagger] = (B^\dagger A^\dagger - A^\dagger B^\dagger)$ eşittir.

Cevap 6)

a) $i(x^2+1)\frac{d}{dx}+ix$ hermityen eşleniğini bulalım.

$$\left(i(x^2+1)\frac{d}{dx}+ix\right)^{\dagger}=\left(-i(x^2+1)\left(\frac{d}{dx}\right)^{\dagger}-ix\right)$$

$$\left(\frac{d}{dx}\right)^{\dagger}=?$$

$$\int \psi^* \frac{d}{dx}\psi\, dx=\int\left\{\frac{d}{dx}(\psi^*\psi)-\frac{d\psi^*}{dx}\psi\right\}dx$$

$$=\int \frac{d}{dx}\psi^*\psi\, dx-\int\frac{d\psi^*}{dx}\psi\, dx$$

$$=\cancel{\psi\psi^*\Big|_{-\infty}^{\infty}}-\int\frac{d\psi^*}{dx}\psi\, dx$$

$$\int \psi^*\frac{d\psi}{dx}dx=\int-\frac{d\psi^*}{dx}\psi\, dx$$

$$\langle\psi|A\psi\rangle=\langle A^{\dagger}\psi|\psi\rangle$$

$$\left(\frac{d}{dx}\right)^{\dagger}=-\frac{d}{dx}$$ hermityen değil

$$\left(\frac{d^2}{dx^2}\right)^{\dagger}=\left[\left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right)\right]^{\dagger}=\left(\frac{d}{dx}\right)^{\dagger}\left(\frac{d}{dx}\right)^{\dagger}$$

$$=\frac{d^2}{dx^2}$$ hermityen

$$\left(i(x^2+1)\frac{d}{dx}+ix\right)^\dagger = \left(-i(x^2+1)\left(\frac{d}{dx}\right)^\dagger - ix\right)$$

$$= -i(x^2+1)\left(-\frac{d}{dx}\right) - ix$$

$$= i(x^2+1)\frac{d}{dx} - ix$$

b) $$\left(i\frac{d^2}{dx^2}+\frac{d}{dx}\right)^\dagger = \left(-i\left(\frac{d^2}{dx^2}\right)^\dagger - \left(\frac{d}{dx}\right)^\dagger\right)$$

$$= \left(-i\frac{d^2}{dx^2} - \left(-\frac{d}{dx}\right)\right)$$

$$= -i\frac{d^2}{dx^2} + \frac{d}{dx}$$

7) $e^{i\theta A} = 1 + i\theta A + \frac{(i\theta A)^2}{2!} + \frac{(i\theta A)^3}{3!} + \frac{(i\theta A)^4}{4!} + \frac{(i\theta A)^5}{5!} + \cdots$

$$= \left(1 - \frac{(\theta A)^2}{2!} + \frac{(\theta A)^4}{4!} - \frac{(\theta A)^6}{6!} + \cdots\right)$$

$$+ \left(i\theta A - \frac{i(\theta A)^3}{3!} + \frac{i(\theta A)^5}{5!} + \cdots\right)$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow AA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \mathbb{1}$$

$$e^{i\theta A} = \left(1 - A^2\frac{\theta^2}{2!} + A^4\frac{\theta^4}{4!} - \frac{A^6\theta^6}{6!} + \cdots\right)$$

$$+ iA\left(\theta - A^2\frac{\theta^3}{3!} + A^4\frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + iA\sin\theta$$

Cevap 8) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

a) $A^{+} = \tilde{A}^{*} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = A$

$A^{+} = A$ olduğundan A matrisi hermityendir.

$$\det(A - \lambda \mathbb{1}) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$$= -\lambda\left[(-1-\lambda)(-\lambda)\right] + 1\left[-(-1-\lambda)\right]$$

$$= -\lambda^2(1+\lambda) + 1 + \lambda = (1+\lambda)(1-\lambda^2)$$

$$\left.\begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = 1 \\ \lambda_3 = -1 \end{array}\right\} \text{özdeğerler}$$

$\lambda_1 = -1$ özdeğeri için

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1^{(1)} \\ a_2^{(1)} \\ a_3^{(1)} \end{pmatrix} = \lambda_1 \begin{pmatrix} a_1^{(1)} \\ a_2^{(1)} \\ a_3^{(1)} \end{pmatrix} = -1 \begin{pmatrix} a_1^{(1)} \\ a_2^{(1)} \\ a_3^{(1)} \end{pmatrix}$$

$$\begin{aligned} a_3^{(1)} &= -a_1^{(1)} \\ -a_2^{(1)} &= -a_2^{(1)} \\ a_1^{(1)} &= -a_3^{(1)} \end{aligned} \Rightarrow \begin{aligned} a_2^{(1)} &= 0 \\ a_1^{(1)} &= -1 \\ a_3^{(1)} &= 1 \end{aligned} \Rightarrow \vec{a}^{(1)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda_2 = 1$ için,

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1^{(2)} \\ a_2^{(2)} \\ a_3^{(2)} \end{pmatrix} = 1 \begin{pmatrix} a_1^{(2)} \\ a_2^{(2)} \\ a_3^{(2)} \end{pmatrix}$$

$$\begin{aligned} a_3^{(2)} &= a_1^{(2)} \\ -a_2^{(2)} &= a_2^{(2)} \\ a_1^{(2)} &= a_3^{(2)} \end{aligned} \Rightarrow \begin{aligned} a_2^{(2)} &= 0 \\ a_1^{(2)} &= 1 \end{aligned} \Rightarrow \vec{a}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$|2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda_3 = -1$ için $|3\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Ortonormalite özelliği , $\langle 1|2\rangle = \delta_{12}$

$1 = 2 \rightarrow \delta_{12} = 1$

$1 \neq 2 \rightarrow \delta_{12} = 0$

$\langle 1|2\rangle = \frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}_{1\times 3}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}_{3\times 1} = \frac{1}{2}(-1+1) = 0$

$\langle 1|3\rangle = \frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}(0+0+0) = 0$

$\langle 2|3\rangle = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}(0+0+0) = 0$

$\langle 1|1\rangle = \frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}(1+1) = 1$

$\langle 2|2\rangle = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}(1+1) = 1$

$\langle 3|3\rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1$

Kapalılık özelliği , $\sum_{a'} |a'\rangle\langle a'| = 1$

$|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| = \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}_{3\times 1}\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}_{1\times 3}$

$+ \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}_{3\times 1}\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}_{1\times 3} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{3\times 1}\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}_{1\times 3}$

$|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}_{3\times 3} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}_{3\times 3} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times 3}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Ortormalite ve kapalılık özellikleri sağlandı.

8b) Projeksiyon operatörleri, $|1\rangle\langle 1|$, $|2\rangle\langle 2|$ ve $|3\rangle\langle 3|$

$$|1\rangle\langle 1| = \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}_{3\times 1}\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}_{1\times 3} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}_{3\times 3}$$

$$|2\rangle\langle 2| = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$|3\rangle\langle 3| = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

8c) $|\alpha\rangle = a|1\rangle + b|2\rangle + c|3\rangle$

$$|\alpha\rangle = \frac{a}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix} + \frac{b}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}-a/\sqrt{2} + b/\sqrt{2}\\ c \\ a/\sqrt{2} + b/\sqrt{2}\end{pmatrix}$$

$$|1\rangle\langle 1|\alpha\rangle = \frac{1}{2}\begin{pmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 1\end{pmatrix}_{3\times 3}\begin{pmatrix}-a/\sqrt{2} + b/\sqrt{2}\\ c \\ a/\sqrt{2} + b/\sqrt{2}\end{pmatrix}_{3\times 1}$$

$$= \frac{1}{2}\begin{pmatrix}-a/\sqrt{2} \cancel{+ b/\sqrt{2}} - a/\sqrt{2} \cancel{- b/\sqrt{2}}\\ 0 \\ a/\sqrt{2} \cancel{- b/\sqrt{2}} + a/\sqrt{2} \cancel{+ b/\sqrt{2}}\end{pmatrix}_{3\times 1}$$

$$= \frac{1}{2}\begin{pmatrix}-2a/\sqrt{2}\\ 0 \\ 2a/\sqrt{2}\end{pmatrix}$$

$$= \frac{a}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$= a|1\rangle$$

//

$$|2\rangle\langle 2|\alpha\rangle = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -a/\sqrt{2} + b/\sqrt{2} \\ c \\ a/\sqrt{2} + b/\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \cancel{-a/\sqrt{2}} + b/\sqrt{2} + \cancel{a/\sqrt{2}} + b/\sqrt{2} \\ 0 \\ \cancel{-a/\sqrt{2}} + b/\sqrt{2} + \cancel{a/\sqrt{2}} + b/\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2b/\sqrt{2} \\ 0 \\ 2b/\sqrt{2} \end{pmatrix}$$

$$= \frac{b}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= b\,|2\rangle$$

$$|3\rangle\langle 3|\alpha\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -a/\sqrt{2} + b/\sqrt{2} \\ c \\ a/\sqrt{2} + b/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ c \\ 0 \end{pmatrix}$$

$$= c\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= c\,|3\rangle$$

8d) $U = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$

$$U^{+} = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$$U^{+} A U = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 - 1/2 & -1/2 + 1/2 & 0 \\ 1/2 - 1/2 & 1/2 + 1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Cevap 9) $H = \hbar w_0 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $B = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

a) Hermityen operatörlerin özdeğerleri gerçel olduğundan hermityen operatörler fiziksel olarak gözlemlenebilirdir.

$$H^{\dagger} = \tilde{H}^{*} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = H$$

H operatörü fiziksel olarak gözlemlenebilir.

$$B^{\dagger} = \tilde{B}^{*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B$$

B operatörü de fiziksel olarak gözlemlenebilir.

b) $|a'\rangle$, H ve B nin ortak özvektörü olsun,

$H|a'\rangle = \hbar w_0 |a'\rangle = E_0 |a'\rangle$ $B|a'\rangle = b|a'\rangle$

$$[H,B]|a'\rangle = (HB - BH)|a'\rangle = HB|a'\rangle - BH|a'\rangle$$
$$= Hb|a'\rangle - BE_0|a'\rangle$$
$$= bE_0|a'\rangle - E_0 b|a'\rangle$$
$$= 0$$

$\rightarrow$

Yani $[H, B] = 0$ ise ortak özvektörleri vardır.

9c) $B = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ operatörü birimsel

matrisinin sütunları orthonormal tabanı belirler;

$\vec{a}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{a}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{a}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

10) $U^{\dagger} = U^{-1} \Longleftrightarrow$ Birimsel operatör

$U^{\dagger} U = \mathbb{1}$

$$\left(\frac{1+iA}{1-iA}\right)^{\dagger}\left(\frac{1+iA}{1-iA}\right) = \left(\frac{1+iA-iA^{\dagger}+A^{\dagger}A}{1+iA^{\dagger}-iA+A^{\dagger}A}\right) = \mathbb{1}$$

$A^{\dagger} = A \Rightarrow \left(\frac{1+iA}{1-iA}\right)$ birimsel operatördür.

b) $$\left(\frac{A+iB}{\sqrt{A^2+B^2}}\right)^{\dagger}\left(\frac{A+iB}{\sqrt{A^2+B^2}}\right) = \left(\frac{A^{\dagger}-iB^{\dagger}}{\sqrt{A^{\dagger}A^{\dagger}+B^{\dagger}B^{\dagger}}}\right)\left(\frac{A+iB}{\sqrt{A^2+B^2}}\right)$$

$$= \frac{A^{\dagger}A + iA^{\dagger}B - iB^{\dagger}A + B^{\dagger}B}{\sqrt{A^{\dagger}A^{\dagger}+B^{\dagger}B^{\dagger}}\sqrt{AA+BB}} \quad , \quad A^{\dagger} = A \ \wedge \ B = B \Rightarrow$$

$$= \frac{A^2 + iAB - iBA + B^2}{\sqrt{A^2+B^2}\sqrt{A^2+B^2}} = \frac{A^2+B^2+i[A,B]}{A^2+B^2}$$

$$= \mathbb{1} \Longleftrightarrow [A,B] = 0$$

Yani $A = A^{\dagger}$ hermityen, $B = B^{\dagger}$ hermityen olmalı ayrıca $[A,B] = 0$ yani A ve B komüt edebilmeli

11) $[x_i, p_j] = i\hbar\,\delta_{ij}$

$$[x, p_x] = xp_x - p_x x = i\hbar \Rightarrow xp_x = p_x x + i\hbar$$
$$p_x x = xp_x - i\hbar$$

$$(xp_x + p_x x)^2 = (xp_x + p_x x)(xp_x + p_x x)$$

$$= xp_x x p_x + xp_x p_x x + p_x x x p_x + p_x x p_x x$$

$$= x(xp_x - i\hbar)p_x + xp_x(xp_x - i\hbar)$$
$$+ p_x x(p_x x + i\hbar) + p_x(p_x x + i\hbar)x$$

$$xp_x(xp_x - i\hbar) = xp_x x p_x - i\hbar x p_x$$
$$= x(xp_x - i\hbar)p_x - i\hbar x p_x$$
$$= x^2 p_x^2 - i\hbar x p_x - i\hbar x p_x$$
$$= x^2 p_x^2 - 2i\hbar x p_x$$

$$p_x x(p_x x + i\hbar) = p_x x p_x x + i\hbar p_x x$$
$$= p_x(p_x x + i\hbar)x + i\hbar p_x x$$
$$= p_x^2 x^2 + 2i\hbar p_x x$$

$$= x^2p_x^2 - i\hbar x p_x + x^2 p_x^2 - 2i\hbar x p_x + p_x^2 x^2 + 2i\hbar p_x x$$
$$+ p_x^2 x^2 + i\hbar p_x x \Rightarrow$$

$$= 2\left(x^2 p_x^2 + p_x^2 x^2\right) - 3i\hbar\, x p_x + 3i\hbar\, p_x x$$

$$= 2\left(x^2 p_x^2 + p_x^2 x^2\right) - 3i\hbar\,(\underbrace{x p_x - p_x x}_{i\hbar})$$

$$= 2\left(x^2 p_x^2 + p_x^2 x^2\right) + 3\hbar^2$$